Mr. Edmunds seeks to prove active intercommunication. Six centuries before the same period, one of the last of the Pharaohs opened a canal from the Nile to the Red Sea to bring his country into communication with the Eastern trade in defiance of her Mesopotamian oppressors. Six centuries after the Christian era Buddhist and Christian legends were so mingled in Western Asia, that the Koran absolutely confused the two; while a little later in Eastern Asia a Chinese emperor issued an edict forbidding the same confusion then prevalent in his dominions.

It should hardly be necessary to recall that Palestine was the West-land of the Mesopotamian civilization just as India was the East-land; and that it was at the western rim of that ancient culture-field, and not from the Greek or Roman environment, that the Christian Gospels arose, just as it was at the eastern rim that the Buddhist writings were formulated. Without in any way assuming identity of origin or purpose, it would be strange indeed if there were not identity of expression and parallelism of thought between these two great Canons; and Mr. Edmunds's proof of that identity is a distinct contribution to human knowledge.

WILFRED H. SCHOFF.

PHILADELPHIA, November, 1911.

MR. BERTRAND RUSSELL'S FIRST WORK ON THE PRINCIPLES OF MATHEMATICS.

In *The Monist* for January, 1910,[1] Dr. Carus has criticized an article of Mr. Bertrand Russell's on "Recent Work on the Principles of Mathematics," published in the *International Monthly* for 1901. A copy of the article lately came into my hands, corrected in Mr. Russell's handwriting back again to what he originally wrote.[2] The editor or type-setter occasionally changed Mr. Russell's words to words which he considered more dignified, perhaps. Thus, the *International Monthly* makes Mr. Russell say[3] that in pure mathematics we "take any hypothesis that seems assuring, and deduce its consequences." Mr. Russell had written "amusing," and the substitution of "assuring" rather took away from the force of Mr. Russell's contention that in mathematics we are not in the least con-

[1] Vol. XX, pp. 46-63.

[2] Mr. Russell has since kindly told me that this statement is correct.

[3] Quoted in *The Monist*, Vol. XX, p. 50.

cerned with the truth or otherwise of our hypotheses or consequents, but merely with the truth of the deductions.

The import of another alteration I quite fail to grasp. Mr. Russell wrote that "pure mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of *anything*, then such and such another proposition is true of that thing." The *International Monthly*[4] put "asseverations" for "assertions"; and so Dr. Carus[5] remarked: "I wish Professor Russell would not describe mathematics as consisting of 'asseverations'; the very idea is jarring on my conception of the nature of mathematics."

When Dr. Carus[6] uses here, as he often has before, the word "anyness" to describe what is the fundamental characteristic of mathematics in his conception, he seems to be in agreement with one of the main tenets of Mr. Russell:[7] the propositions of logic "can be put into a form in which they apply to anything whatever"; "we never know what [which thing] we are talking about" in mathematics; the assertions are that, "if such and such a proposition is true of *anything*, then such and such another proposition is true of that thing."

I am going to try shortly to explain Mr. Russell to my readers. Mr. Russell's work on the principles of mathematics and the relation of mathematics to logic "is by no means," as Couturat said,[8] "like certain philosophical systems in fashion, a brilliant paradox, an individual and ephemeral fantasy, without roots in the past and without fruits in the future, but the necessary culmination and crowning of all the critical researches to which some mathematicians have given themselves up for the last half-century. It is a well-known fact that modern mathematics have constantly tended to deductive rigor of the reasonings and logical purity of the concepts. To these new needs of the scientific spirit a logic more and more exact and refined had to respond; the indispensable instrument of this new logic is the 'symbolic[9] logic' invented by Peano, practised by a whole school of mathematicians, and perfected by Russell.

[4] Quoted in *The Monist*, Vol. XX, p. 50.

[5] *Ibid.*, p. 53.

[6] *Ibid.*, p. 50.

[7] *Ibid.*, pp. 47, 49, 50.

[8] *Les Principes des mathématiques*, Paris, 1905, pp. v-vi. A translation of Couturat's work by the author of this article is in preparation.

[9] As a matter of fact, Peano has always called his system "mathematical logic." The name of Frege ought to be mentioned with Peano's in this connection.

It is owing to this *logistics* (as we will call it) that all mathematical theories have become susceptible of being subjected to a precise and subtle analysis, and of being reconstructed logically with a small number of fundamental data (primitive principles and notions). It is owing to this that Russell has been able, while completing on certain points this work of logical reduction, to systematize all the results acquired in a vast and profound synthesis, which is the quintessence of preceding works, and which manifests the spirit of modern mathematics."

Consider, for a moment, what this logical analysis means. Take the science of arithmetic. All its material and principles have to be reduced to logical terms and expressed unambiguously. This enormously important work is extraordinarily long and often tedious. Processes of thought that most mathematicians perform more or less accurately by "intuition" often take up, in expression, pages of symbols of logical deduction—if such deduction is possible; but then we get complete, and not only "moral," certainty, and an insight into the structure of certain truths. In Dr. Whitehead and Mr. Russell's latest book[10] there are 666 pages, most of them written in symbols, often with abbreviated proofs, and yet the definition of numbers is not yet reached! Things called "1" and "2" are defined, but not till the second volume will it appear that they are numbers!

There is a story current in Cambridge that, after a term's lecturing on the principles of mathematics, Mr. Russell informed his hearers that if they were good they should do simple addition next term....And so recently as 1888 Dedekind's tract of 58 pages, *Was sind und was sollen die Zahlen?*[11] was derided by some mathematicians because it devoted so much space to the foundations of arithmetic!

Few people can see the immense importance of Mr. Russell's work; fewer know how laborious it has been and by what splendid qualities of mind and character it has been inspired. That is all I can say on this head, as I do not wish to gush and am not writing an obituary notice. Not quite so few people know how brilliant Mr. Russell's work is. Mr. Russell's investigations have revealed some very striking things, and Mr. Russell has said them strikingly —said them, too, in books and articles which are read with delight, and sometimes with profit, by those who are untrained to follow

[10] *Principia Mathematica*, Vol. I, Cambridge, 1910.

[11] English translation by W. W. Beman, in Dedekind's *Essays on the Theory of Numbers*, Chicago, 1901.

Mr. Russell's work. I suppose Mr. Russell has a natural love of paradox, but his paradox is always used to give point to the statement of some truth. In his talk and writings, Mr. Russell is conscientious, truth-loving, keen and witty.

I now propose to analyze the *International Monthly* article and to try to show how the fundamental doctrines of the *Principles of Mathematics* are shortly stated in it. This will continue my article in *The Monist* for January, 1910;[12] and in future I hope to trace Mr. Russell's work beyond 1903.

I.

The first published indication of the effect of Peano's work on Russell appeared in an article by Russell on "Recent Work on the Principles of Mathematics" in the *International Monthly* for 1901.[13] Boole, he said,[14] was "mistaken in supposing that he was dealing with the laws of thought: the question how people actually think was quite irrelevant to him,....His book was in fact concerned with formal logic, and this is the same thing as mathematics." Then came[15] a definition of pure mathematics: "Pure mathematics consists entirely of assertions to the effect that if such and such a proposition is true of *anything*, then such and such a proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is of which it is supposed to be true. Both these points would belong to applied mathematics. We start, in pure mathematics, from certain rules of inference, by which we can infer that *if* one proposition is true, then so is some other proposition. These rules of inference constitute the principles of formal logic. We then take any hypothesis that seems amusing, and deduce its consequences. *If* our hypothesis is about *anything*, and not about some one or more particular things, then our deductions constitute mathematics. Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."

The reduction of mathematics to logic was spoken of:[16] "Now the fact is that, though there are indefinables and indemonstrables in every branch of applied mathematics, there are none in pure

[12] Vol. XX, pp. 93-118.

[13] Vol. IV, pp. 83-101.

[14] *Ibid.*, p. 83.

[15] *Ibid.*, pp. 83-84. For "assertions" was misprinted "asseverations," and for "amusing" was misprinted "assuring."

[16] *Ibid.*, p. 84.

mathematics except such as belong to general logic. Logic, broadly speaking, is distinguished by the fact that its propositions can be put into a form in which they apply to anything whatever. All pure mathematics—arithmetic, analysis, and geometry—is built up by combinations of the primitive ideas of logic, and its propositions are deduced from the general axioms of logic, such as the syllogism and the other rules of inference."

When dealing with questions of the principles of mathematics, the function of symbolism is exactly the opposite to that of symbolism in the other parts of mathematics. Russell said:[17] "The fact is that symbolism is useful because it makes things difficult. (This is not true of the advanced parts of mathematics, but only of the beginnings.) What we wish to know is, what can be deduced from what. Now, in the beginnings, everything is self-evident; and it is very hard to see whether one self-evident proposition follows from another or not. Obviousness is always the enemy of correctness. Hence we invent some new and difficult symbolism, in which nothing seems obvious. Then we set up certain rules for operating on the symbols, and the whole thing becomes mechanical. In this way we find out what must be taken as premise and what can be demonstrated or defined."

II.

Referring to Peano's three indefinables in arithmetic, Russell remarked:[18] "Even these three can be explained by means of the notions of *relation* and *class*; but this requires the logic of relations which Professor Peano has never taken up."

Russell[19] then indicated his contradiction:

"There is a greatest of all infinite [cardinal] numbers, which is the number of all things altogether, of every sort and kind. It is obvious that there cannot be a greater number than this, because, if everything has been taken, there is nothing left to add. Cantor has a proof that there is no greater number, and if this proof were valid, the contradictions of infinity would re-appear in a sublimated form. But on this one point, the master has been guilty of a very subtle fallacy, which I hope to explain in some future work."

* * *

Russell's statement of Zeno's puzzle about Achilles and the tortoise was:[20]

[17] *Ibid.*, pp. 85-86. [18] *Ibid.*, p. 87. [19] *Ibid.*, p. 95.
[20] *Ibid.*, pp. 95-96.

"The argument is this: Let Achilles and the tortoise start along a road at the same time, the tortoise (as is only fair) being allowed a handicap. Let Achilles go twice as fast as the tortoise, or ten times or a hundred times as fast. Then he will never reach the tortoise. For at every moment the tortoise is somewhere, and Achilles is somewhere; and neither is ever twice in the same place while the race is going on. Thus the tortoise goes to just as many places as Achilles does, because each is in one place at one moment, and in another at any other moment. But if Achilles were to catch up with the tortoise the places where the tortoise would have been would be only part of the places where Achilles would have been. Here, we must suppose, Zeno appealed to the maxim that the whole has more terms than the part. Thus, if Achilles were to overtake the tortoise, he would have been in more places than the tortoise; but we saw that he must, in any period, be in exactly as many places as the tortoise. Hence we infer that he can never catch the tortoise. This argument is strictly correct if we allow the axiom that the whole has more terms than the part. As the conclusion is absurd, the axiom must be rejected, and then all goes well. But there is no good word to be said for the philosophers of the past two thousand years and more, who have all allowed the axiom and denied the conclusion."

* * *

The converse of the Achilles, which Russell called "the paradox of Tristram Shandy," was then described;[21] and the remark was made[22] that the notion of continuity depends upon that of *order,* and that "nowadays, quantity is banished altogether [from mathematics] except from one little corner of geometry, while order more and more reigns supreme." Nowadays, too, a limit is defined ordinally.[23]

Then:[24] "Geometry, like arithmetic, has been subsumed in recent times under the general study of order. It was formerly supposed that geometry was the study of the nature of the space in which we live, and accordingly it was urged by those who held that what exists can only be known empirically, that geometry should really be regarded as belonging to applied mathematics. But it has gradually appeared, by the increase of non-Euclidean systems, that geometry throws no more light upon the nature of space than arithmetic

[21] *Ibid.,* pp. 96-97.

[22] *Ibid.,* p. 97.

[23] *Ibid.,* pp. 97-98.

[24] *Ibid.,* p. 98.

throws upon the population of the United States. Geometry is a whole collection of deductive sciences based on a corresponding collection of sets of axioms. One set of axioms is Euclid's; other equally good sets of axioms lead to other results. Whether Euclid's axioms are true, is a question as to which the pure mathematician is indifferent; and what is more, it is a question which it is theoretically impossible to answer with certainty in the affirmative. It might possibly be shown, by very careful measurements, that Euclid's axioms are false; but no measurements could ever assure us (owing to the errors of observation) that they are exactly true. Thus the geometer leaves to the man of science to decide, as best he may, what axioms are most nearly true in the actual world. The geometer takes any set of axioms that seem interesting, and deduces their consequences. What defines geometry, in this sense, is that the axioms must give rise to a series of more than one dimension. And it is thus that geometry becomes a department in the study of order."

Russell[25] then shortly dealt with the methods used by Peano and Fano in geometry, and finally[26] remarked that "the proof that all pure mathematics, including geometry, is nothing but formal logic, is a fatal blow to the Kantian philosophy."

III.

Let us now point out how this popular article gives indications of his logical work up to 1903.

To begin with, the two great influences on Russell's mathematical and logical work were Georg Cantor and Peano. Cantor had, in 1895 and 1897,[27] brought his researches on transfinite numbers and ordinal types to a close by two articles in which the principles of the subject were stated in an almost perfect logical form. Obviously, the whole question threw a great and welcome light on the principles of arithmetic.[28] Peano invented a symbolic logic which was especially adapted to the analysis and expression of mathematical theories. But Peano's logic was incomplete. It neglected the logic of relations, which was founded and developed by De Morgan, C. S. Peirce, and Schröder; and only contained a symbolical expression of the theory—unused, by the way, in Peano's symbolic

[25] *Ibid.*, pp. 99-100.

[26] *Ibid.*, p. 101.

[27] *Mathematische Annalen*, Vols. XLVI and XLIX. An annotated translation of these articles by the author is in preparation.

[28] Cf. my article on "Transfinite Numbers and the Principles of Mathematics" in *The Monist* for January, 1910.

exposition of arithmetic—of the "representations" of Richard Dedekind.[29] The logic of relations was, as Schröder had observed, necessary for the translation of Cantor's conceptions and proofs into a symbolic (speaking technically) form; and it was necessary in order to complete Peano's theory of arithmetic by defining in logical terms the three indefinables referred to above. Russell completed Peano's logic by a logic of relations in which the Peirce-Schröder ideas were modified so as to fit in with a logic which comprised more subtle distinctions than that of Schröder, in two papers, "Sur la logique des relations, avec des applications à la théorie des séries," and "Théorie des séries bien-ordonnées," which were published in Peano's *Revue de Mathématiques* for 1902,[30] and of the first of which an account was given in Russell's *Principles of Mathematics* of 1903.[31] The logic of relations gave to Russell the means of defining Peano's indefinables of arithmetic, and of proving his primitive propositions of arithmetic.[32]

Peano had emphasized that it was the notion of implication between propositions containing variables—or, as Russell expressed it, of *formal* implications[33] between propositional *functions*,[34] and not implication between (constant) propositions, that is used in mathematics. Further, the development of non-Euclidean geometry had shown in the most striking manner that, in pure mathematics, as in formal logic, we are not concerned with the truth or otherwise of the hypotheses. "Until the nineteenth century," said Russell,[35] "geometry meant Euclidean geometry, *i. e.*, a certain system of propositions deduced from premises supposed to describe the space in which we live....," but now, owing to investigations with premises other than Euclid's, "geometry has become....a subject in which the assertions are that such and such consequences follow from such and such premises, not that entities such as the premises describe actually exist." And all this goes some way

[29] Cf. the English translation of Dedekind's pamphlet in Dedekind's *Essays on the Theory of Numbers*, Chicago, 1901.

[30] An account of Peano's and Russell's logic was given by A. N. Whitehead in his paper "On Cardinal Numbers" in the *Amer. Journal of Math.*, Vol. XXIV, 1902, pp. 367-394.

[31] *The Principles of Mathematics*, Vol. I [the *Principia Mathematica* of Whitehead and Russell, of which the first volume was published in 1910, takes the place of the second volume], pp. 23-26; cf. Couturat, *op. cit.*, pp. 27-34.

[32] *Principles*, pp. 124-128.

[33] *Ibid.*, pp. 5, 11, 14, 36-41; Couturat, *op. cit.*, pp. 4, 21.

[34] *Principles*, pp. 13, 19; Couturat, *op. cit.*, p. 17.

[35] *Principles*, pp. 372-373.

towards explaining the definition of pure mathematics with which Russell's book begins:

"Pure mathematics is the class of all propositions of the form 'p implies q,' where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants. And logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member, the notion of *such that,* the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics *uses* a notion which is not a constituent of the propositions which it considers, namely the notion of truth."

In this definition culminates the discovery contributed to by Leibniz, Frege, Dedekind, Schröder, and a host of others, that pure mathematics is logic and logic alone. Hence Russell's[36] anti-Kantianism.

* * *

In the question of infinity, we have a discussion of Zeno's puzzles,[37] and meet again the paradox of Tristram Shandy.[38] When discussing continuity, Russell[39] made more explicit Cantor's discovery (1895) that it is a purely ordinal notion; and then, too, Russell succeeded in maintaining his theses that the theory of limits is purely ordinal,[40] that geometry is the study of order,[41] and that the notion of quantity is superfluous in mathematics.[42]

* * *

Finally we come to Russell's[43] contradiction. Starting from a study of Cantor's proof of 1892 that there is no greatest cardinal number, Russell discovered a very simple argument: If w denotes the class of all those entities x such that x is not a member of x; then, obviously, if w is a member of w, w is not a member of w, while if w is not a member of w, w is a member of w. This contra-

[36] *Principles,* pp. 4, 158, 259, 373, 442, 456-461; Couturat, *op. cit.,* pp. 235-308.

[37] *Principles,* pp. 347-353, 358-360.

[38] *Ibid.,* pp. 358-360.

[39] *Ibid.,* pp. 296-303; Couturat, *op. cit.,* pp. 91-97.

[40] *Principles,* pp. 276-277.

[41] *Ibid.,* p. 372; Couturat, *op. cit.,* p. 134.

[42] *Principles,* p. 158; Couturat, *op. cit.,* p. 98.

[43] *Principles,* pp. 364-368, 101-107.

diction, which threw doubt upon the legitimacy of the concept of class, and hence upon that of the science of arithmetic, showed itself as allied in principle to the paradoxes in the theory of aggregates discovered by Burali-Forti, König, Richard, and others, and to the old logical difficulty about the Cretan who said that Cretans were liars, and was only satisfactorily solved by Russell in 1905. Of this more elsewhere.

It only remains at present to refer to the work of Frege. He did his magnificent work on the principles of logic and mathematics alone and almost too independently, and his subtle distinctions and acute analysis have had great influence on modern work. But at first Russell had hardly heard of him, and re-discovered for himself many of his distinctions and views. In his *Principles,*[44] Russell devoted many pages to a careful critical estimate of Frege's work. I hope to give an account of Frege's work later.

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ALFRED BINET.*

OBITUARY.

Readers of *The Monist* are well acquainted with the name of Alfred Binet. That eminent psychologist died at Paris October 18, 1911, at the age of 54, from an attack of cerebral apoplexy. He was born at Nice, July 11, 1857. He first took up the study of law, but later turned his attention to natural sciences, and finally directed all his efforts to psychology. In 1894 in collaboration with Beaunis at the laboratory of physiological psychology of the Sorbonne, he founded the *Année psychologique,* an important publication of permanent value.

His principle works are *Vie psychique des micro-organismes* (English edition, *The Psychic Life of Micro-Organisms,* Open Court Publishing Co., 1894); *Psychologie du raisonnement* (English edition, *The Psychology of Reasoning,* Open Court Publishing Co., 1899); *Le magnetisme animal, Les alterations de la personnalité, Psychologie des grands calculateurs et joueurs d'échechs, Etude experimentale de l'intelligence, L'âme et le corps.* To these we should also add a number of articles on an equal variety of subjects, capil-

[44] Pp. 501-522.

* Translated for *The Monist.*